Unit-3 : Constraints and their classifications, Lagrange's equation of motion for holonomic system, Gibbs-Appell's principle of least constraint, Work energy relation for constraint forces of shielding friction. 20L

## Course : BMH6PW01

## Project Work (Marks : 75)

Any student may choose Project Work in place of one Discipline Specific Elective (DSE) paper of Semester -VI. Project Work will be done considering any topic on Mathematics and its Applications. The marks distribution of the Project work is 40 Marks for written submission, 20 Marks for Seminar presentation and 15 Marks for Viva-Voce.

# ABHEDANANDA MAHAVIDYALAYA

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Ref. No. ..........................                                                    Date:

## COMPLETION CERTIFICATE

This is to certify that the following students of semester VI (Hons.) have successfully completed his/her Project Work under the supervision of **Prof. Chhatu Manuel Mardi**, Department of Mathematics, during the academic year 2023-24.

| SI NO | Roll NO | Name of the student | Title of the Project Work |
|---|---|---|---|
| 1 | 210330100017 | DIP MONDAL | Game theory and it's application |
| 2 | 210330100005 | AHAMMAD HOSSAIN | "Vam" (linear programming) |
| 3 | 210330100014 | BISWARUP MUKHERJEE | A special case of transportation problem |
| 4 | 210330100034 | RAJLAKSHI CHANDRA | Assignment problem of linear programming |
| 5 | 210330100035 | RUPAK KHANDAIT | Vogel's approx methods |

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| Sl No | Roll No | Name of the student | Title of the Project Work |
|---|---|---|---|
| 1. | 210330100001 | ABDUL HASIM | Ring |
| 2 | 210330100028 | PRANOBENDU ADHIKARI | Ring and it's properties |
| 3 | 210330100004 | ABHISHEK BASAK | Ring theory |
| 4 | 210330100012 | BIKRAM PAL | Graphical method of linear programming |

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| Sl NO | Roll NO | Name of the student | Title of the Project Work |
|---|---|---|---|
| 1 | 210330100008 | ANKUR BANDYOPADHYAY | Graphical method of LPP |
| 2 | 210330100002 | ABHIJIT MONDAL | "Simplex method" (linear programming) |
| 3 | 210330100045 | SHAMIT KUMAR MAJHI | Transportation problem: a linear programming approach |
| 4 | 210330100058 | SUJATA GHOSH | Assignment problem of linear programming |
| 5 | 210330100061 | SURAJIT DAS | LPP transportation problem |

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## COMPLETION CERTIFICATE

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| SI NO | Roll NO | Name of the student | Title of the Project Work |
|---|---|---|---|
| 1 | 210330100047 | SHREYA ROY | Classification of partial differential equation |
| 2 | 210330100054 | SUBHAMOY BHATTACHARYA | Algebraic and trancendental equations |
| 3 | 210330100060 | SUMAN MANDAL | Application of derivatives |
| 4 | 210330100031 | PUJA SAHA | Convolution operation on image processing with the help of matrix |

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# THE UNIVERSITY OF BURDWAN

## ABHEDANANDA MAHAVIDYALAYA

SEMISTER : VI

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**REGISTRATION NO:- 202101026530 of 2021-22**

**SUBJECT: MATHEMATICS**

**Under the guidance of Prof Dr. SUDIPTA SENAPATI**

**A project work presented for the degree of Bachelor of Science**

**PROJECT TOPIC: TRANSPORTATION PROBLEM: A LINEAR PROGRAMMING APPROACH**

# TRANSPORTATION PROBLEM: A LINEAR PROGRAMMING APPROACH

Submitted by

SHAMIT KUMAR MAJHI

University Roll No: 210330100045

Registration No: 202101026530 of 2021-2022

Associated Professor

Head of Department of Mathematics

(Supervisor)

# ACKNOWLEDGEMENT

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Also, I would like to thank all of my teacher Surya Kanta Mondal, Chhatu Manuel Mardi for allowing me to work on this project and their co-operation.

Last but not the least I would like to thank my parents, brother and friends for their blessings and inspiration.

Shamit Kumar Majhi
SHAMIT KUMAR MAJHI

Date: 26/07/2024
Sem: VI
Mathematics Honours

# CERTIFICATE

This is certify that the project work entitled "Transportation problem: A Linear Programming Approach" is the investigatory project work in mathematics, successfully completed by **SHAMIT KUMAR MAJHI**, student of B.Sc. semester VI (Department Of Mathematics), Abhedananda Mahavidyalaya, under the University Of Burdwan, bearing University Roll No. 210330100045, Registration No: 202101026530 Of 2021-22, under the guidance of Dr. Sudipta Senapati for the partial fulfilment of requirements for the course completion in pursuance.

Date: 26/07/2024

26/7/24

Associate Professor

(Dept. of Mathematics)

# CONTENT

# OBJECTIVES

After studying this chapter, we should be able to

1) Recognize and formulate a transportation problem involving a large number of shipping routes.
2) Drive initial feasible solution using several methods.
3) Drive optimal solution by using Modified Distribution Method.
4) Handle the problem of degenerate and unbalanced transportation problem.
5) Examine multiple optimal solutions, and prohibited routes in the transportation problem.
6) Construct the initial transportation table for a trans-shipment problem.
7) Solve a profit maximization transportation problem using suitable changes in the transportation algorithm.

# ABSTRACT

The transportation problem (TP) is a unique kind of Linear Programming Problem (LPP) that handles the division of individual item (finished or raw) from different sources of resource to different destination of need in such a manner that the entire transportation cost is minimized. This project presents the mathematical structure for the transportation problem. It's desirable to decide a transportation schedule which is going to satisfy the foundation availabilities, non-negative restrictions and destination requirements while minimizing the entire cost of transportation. The linear mathematical structure of the transportation problem (MOTP) is a unique sort of linear programming problem where constraints are of uniformity type and the objectives are conflicting with one another. The exciting solution methodology of this problem can be partitioned into two classes. First class consist those that are producing all the sets of effective solution and the second classification speaks to the techniques that are looking for the best compromise solution among the arrangement of proficient solution.

# INTRODUCTION

One important application of linear programming is in the area of physical distribution (transportation) of goods and services from several supply centres to several demand centres. A transportation problem when expressed in terms of an LP model can also be solved by the simplex method. However a transportation problem involves a large number of variable and constraints, solving it using simplex methods takes a long time. Transportation algorithms, namely the MODI (modified distribution) Method have been developed for solving a transportation problem.

The structure of transportation problem involves a large number of shipping routes from several supply centres to several demands centres. Thus, objective is to determine shipping routes between supply centres and demand centres in order to satisfy the required quantity of goods or services at each destination centre, with available quantity of goods or services at each supply centre at the minimum transportation cost and/ or time.

The transportation algorithm help to minimize the total cost of transporting a homogeneous commodity (product) from supply centres to demand centres. However, it can also be applied to the maximization of total value of utility.

There are various types of transportation models and the simplest of them was first presented by F L Hitchcock (1941). It was further developed by T C koopmans (1949) and G B Dantzig (1951). Several extensions of transportation models and methods have been subsequently developed.

# Mathematical model of Transportation problem

Let us consider to illustrate the mathematical model formulation of transportation problem of transporting a single commodity from three sources of supply to four demand destinations. The sources of supply are production, facilities, warehouses or supply centres, each having certain amount of commodity to supply. The destinations are consumption facilities, warehouses or demand centres each having certain amount of requirement (or demand) of the commodity.

**Example:** A company has three production facilities, $S_1$, $S_2$ and $S_3$ with production capacity of 7,9 and 18 units (in 100s) per week of a product respectively. Those units are to be shipped to four warehouse $D_1$, $D_2$ $D_3$ and $D_4$ with requirement of 5,6,7 and 14 units (in 100s) per week, respectively. The transportation costs (in rupees) per unit between factories to warehouse are given in the table below:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Supply |
|---|---|---|---|---|---|
| $S_1$ | 19 | 30 | 50 | 10 | 7 |
| $S_2$ | 70 | 30 | 40 | 60 | 9 |
| $S_3$ | 40 | 8 | 70 | 20 | 18 |
| Demand | 5 | 8 | 7 | 14 | |

Formulate this transportation problem as an LP model to minimize the total transportation cost.

**Model formulation:** Let, $x_{ij}$ = Number of units of the product to be transported from a production facility i (i =1,2,3) to a warehouse j (j=1,2,3). The transportation problem is stated as an LP model as follows:

Minimize (Total Transportation Cost) $Z = 19x_{11} + 30x_{12} + 50x_{13} + 10x_{14} + 70x_{21} + 30x_{22} + 40x_{23} + 60x_{24} + 40x_{31} + 8x_{32} + 70x_{33} + 24x_{34}$

Subject to constraints,

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} = 7 \\ x_{21} + x_{22} + x_{23} + x_{24} = 9 \\ x_{31} + x_{32} + x_{33} + x_{34} = 18 \end{aligned}\right\} \text{(Supply)}$$

$$\left.\begin{aligned} x_{11} + x_{21} + x_{31} = 5 \\ x_{12} + x_{22} + x_{32} = 8 \\ x_{13} + x_{23} + x_{33} = 7 \\ x_{14} + x_{24} + x_{34} = 14 \end{aligned}\right\} \text{(Demand)}$$

and $x_{ij} \geq 0$ for, i = 1,2,3 and j 1,2,3,4

In the above LP model, there are m×n = 3×4 = 12 decision variables, $x_{ij}$ and m+n = 7 constraints, where m are the number of rows and n are the number of columns in a general transportation table.

## General mathematical model of Transportation problem

Let, there be m sources of supply $S_1$, $S_2$ ... $S_m$ having $a_i$ (i=1,2...m) units of supply, respectively to be transported to n destination $D_1$, $D_2$ ...$D_n$ with $b_j$ (j=1,2...n) units of demand, respectively. Let $C_{ij}$ be the cost of shipping one unit of the commodity from source i to destination j. If $x_{ij}$ represent number of unit shipped from source i to destination j. The problem is to determine the transportation schedule so as to minimize the total transportation cost while satisfying the supply and demand condition. Mathematically, the transportation problem in general, may be stated as follows:

Minimize (total cost) $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to the constraints,

$\sum_{j=1}^{n} x_{ij} = a_i$ ; i = 1,2...m (supply constraints)

$\sum_{i=1}^{m} x_{ij} = b_j$ ; j = 1,2...n (demand constraints)

$x_{ij} \geq 0$ for all i and j

**Existence of feasible solution:** A necessary and sufficient condition for a feasible solution to the transportation problem is:

Total supply = Total demand

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

This type of transportation problem is balanced transportation problem.

The general transportation table is given below:

| | $D_1$ | $D_2$ | ... | $D_n$ | Supply ($a_i$) |
|---|---|---|---|---|---|
| $S_1$ | $C_{11}$ ($X_{11}$) | $C_{12}$ ($X_{12}$) | ... | $C_{1n}$ ($X_{1n}$) | $a_1$ |
| $S_2$ | $C_{21}$ ($X_{21}$) | $C_{22}$ ($X_{22}$) | ... | $C_{2n}$ ($X_{2n}$) | $a_2$ |
| . . . | . . . | . . . | | . . . | . . . |
| $S_m$ | $C_{m1}$ ($X_{m1}$) | $C_{m2}$ ($X_{m2}$) | ... | $C_{mn}$ ($X_{mn}$) | $a_m$ |
| Demand ($b_j$) | $b_1$ | $b_2$ | ... | $b_n$ | $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ |

# Method of finding initial basic feasible solution

There are several methods of finding initial basic solution. The methods to be discussed here are:

1) NORTH-WEST CORNER METHOD.

2) ROW-MINIMA METHOD.

3) COLUMN-MINIMA METHOD.

4) MATRIX-MINIMA METHOD.

5) VOGEL'S APPROXIMATION METHOD (VAM).

The initial solution obtained by any of the five method must satisfy the following condition:

i) The solution must be feasible i.e. it must satisfy all the supply and demand constraints (also called rim condition).

ii) The number of positive allocation must be equal to m+n-1, Where m is the number of rows and n is the number of column.

Now the above methods are discussed with illustrations.

## 1) NORTH-WEST CORNER METHOD:

*Step-1:*

Start with the cell at upper left (North-West) corner of the transportation table (or matrix) and allocate commodity equal to minimum of rim values for the first row and first column, i.e. min $(a_1,b_1)$

*Step-2:*

a) If allocate made in step-1 is equal to the supply available at first source ($a_1$, in first row) then move vertically down to the cell (2,1), i.e. second row and first column. Apply step-1 again for next allocation.

b) If allocation made in step-1 is equal to the demand of the first destination ($b_1$, in first column) then move horizontally to the cell (1,2), i.e. first row and second column. Apply step-1 again for next allocation.

c) $a_1 = b_1$, allocate $x_{11} = a_1$ or $b_1$ and move diagonally to the cell (2,2).

<u>*Step-3:*</u>

Continue the procedure step by step till an allocation is made in the south-east corner cell of the transportation table.

**Application:** We solve the transportation problem and find the basic feasible solution using by the North-West corner Method.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $a_i$ |
|---|---|---|---|---|---|---|
| $S_1$ | 2 | 11 | 10 | 3 | 7 | 4 |
| $S_2$ | 1 | 4 | 7 | 2 | 1 | 8 |
| $S_3$ | 3 | 9 | 4 | 8 | 12 | 9 |
| $b_j$ | 3 | 3 | 4 | 5 | 6 | |

**Solution:**

Here, $\sum a_i = \sum b_j = 21$

So, It is a balanced transportation problem.

**Tableau-1**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $a_i$ |
|---|---|---|---|---|---|---|
| $S_1$ | 2 (3) | 11 | 10 | 3 | 7 | 4 |
| $S_2$ | 1 | 4 | 7 | 2 | 1 | 8 |
| $S_3$ | 3 | 9 | 4 | 8 | 12 | 9 |
| $b_j$ | 3 | 3 | 4 | 5 | 6 | |

Allocation at (1,1), cell $x_{11}$ = min ($a_1$,$b_1$) = min (4,3) = 3

So, we deleted $D_1$ Column.

**Tableau-2**

| | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 11 (1) | 10 | 3 | 7 | 1 |
| $S_2$ | 4 | 7 | 2 | 1 | 8 |
| $S_3$ | 9 | 4 | 8 | 12 | 9 |
| $b_j$ | 3 | 4 | 5 | 6 | |

Allocation at (1,2), cell $x_{12} = \min(a_1, b_2) = \min(1,3) = 1$

So, we deleted $S_1$ Row.

**Tableau-3**

| | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $a_i$ |
|---|---|---|---|---|---|
| $S_2$ | 4 (2) | 7 | 2 | 1 | 8 |
| $S_3$ | 9 | 4 | 8 | 12 | 9 |
| $b_j$ | 2 | 4 | 5 | 6 | |

Allocation at (2,2), cell $x_{22} = \min(a_2, b_2) = \min(8,2) = 2$

So, we deleted $D_2$ Column.

**Tableau-4**

| | $D_3$ | $D_4$ | $D_5$ | $a_i$ |
|---|---|---|---|---|
| $S_2$ | 7 (4) | 2 | 1 | 6 |
| $S_3$ | 4 | 8 | 12 | 9 |
| $b_j$ | 4 | 5 | 6 | |

Allocation at (2,3), cell $x_{23} = \min(a_2, b_3) = \min(6,4) = 4$

So, we deleted $D_3$ Column.

**Tableau-5**

| | $D_4$ | $D_5$ | $a_i$ |
|---|---|---|---|
| $S_2$ | 2 ② | 1 | 2 |
| $S_3$ | 8 | 12 | 9 |
| $b_j$ | 5 | 6 | |

Allocation at (2,4), cell $x_{24} = \min(a_2, b_4) = \min(2,5) = 2$

So, we deleted $S_2$ Row.

**Tableau-6**

| | $D_4$ | $D_5$ | $a_i$ |
|---|---|---|---|
| $S_3$ | 8 ③ | 12 | 9 |
| $b_j$ | 3 | 6 | |

Allocation at (3,4), cell $x_{34} = \min(a_3, b_4) = \min(9,3) = 3$

So, we deleted $D_4$ Column.

**Tableau-7**

| | $D_5$ | $a_i$ |
|---|---|---|
| $S_3$ | 12 (6) | 6 |
| $b_j$ | 6 | |

Allocation at (3,5), cell $x_{35}$ = min ($a_3$,$b_5$) = min (6,6) = 6

Now, the final tableau as follows:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $a_i$ |
|---|---|---|---|---|---|---|
| $S_1$ | 2 (3) | 11 (1) | 10 | 3 | 7 | 4 |
| $S_2$ | 1 | 4 (2) | 7 (4) | 2 (2) | 1 | 8 |
| $S_3$ | 3 | 9 | 4 | 8 (3) | 12 (6) | 9 |
| $b_j$ | 3 | 3 | 4 | 5 | 6 | |

Thus, The basic feasible solution is,

$X_{11} = 3$, $x_{12} = 1$, $x_{22} = 2$, $x_{23} = 4$, $x_{24} = 2$, $x_{34} = 3$, $x_{35} = 6$

The cost corresponding to this feasible solution

$= 3\times2 + 1\times11 + 2\times4 + 4\times7 + 2\times2 + 3\times8 + 6\times12$

$= 6+11+8+28+4+24+72$

$= 153$

Total number of variables = m+n-1 = 3+5-1 = 7.

## 2) ROW-MINIMA METHOD:

In this method, we first consider the first row and find the minimum cost cell. Let, (1,1) cell be the cell in the first row with minimum cost.

**Application:** We solve the transportation problem and find the basic feasible solution using by the Row-minima method.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 7 | 10 | 14 | 8 | 30 |
| $S_2$ | 7 | 11 | 12 | 6 | 40 |
| $S_3$ | 5 | 8 | 15 | 9 | 30 |
| $b_j$ | 20 | 20 | 25 | 35 | |

## Solution:

Here, $\sum a_i = \sum b_j = 100$

So, It is a balanced transportation problem.

**Tableau-1**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 7 (20) | 10 | 14 | 8 | 30 |
| $S_2$ | 7 | 11 | 12 | 6 | 40 |
| $S_3$ | 5 | 8 | 15 | 9 | 30 |
| $b_j$ | 20 | 20 | 25 | 35 | |

In the $S_1$ Row the minimum cost cell is (1,1)

So, $x_{11} = \min(a_1, b_1) = \min(30, 20) = 20$

So, we deleted $D_1$ Column.

**Tableau-2**

| | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|
| $S_1$ | 10 | 14 | 8 (10) | 10 |
| $S_2$ | 11 | 12 | 6 | 40 |
| $S_3$ | 8 | 15 | 9 | 30 |
| $b_j$ | 20 | 25 | 35 | |

In the $S_1$ Row the minimum cost cell is (1,4)

So, $x_{14}$ = min ($a_1$,$b_4$) = min (10,35) =10

So, we deleted $S_1$ Row.

**Tableau-3**

| | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|
| $S_2$ | 11 | 12 | 6 (25) | 40 |
| $S_3$ | 8 | 15 | 9 | 30 |
| $b_j$ | 20 | 25 | 25 | |

In the $S_2$ Row the minimum cost cell is (2,4)

So, $x_{24}$ = min ($a_2$,$b_4$) = min (40,25) =25

So, we deleted $D_4$ Column.

**Tableau-4**

| | $D_2$ | $D_3$ | $a_i$ |
|---|---|---|---|
| $S_2$ | 11 (15) | 12 | 15 |
| $S_3$ | 8 | 15 | 30 |
| $b_j$ | 20 | 25 | |

In the $S_2$ Row the minimum cost cell is (2,2)

So, $x_{22} = \min(a_2, b_2) = \min(15,20) = 15$

So, we deleted $S_2$ Row.

**Tableau-5**

| | $D_2$ | $D_3$ | $a_i$ |
|---|---|---|---|
| $S_3$ | 8 (5) | 15 | 30 |
| $b_j$ | 5 | 25 | |

In the $S_3$ Row the minimum cost cell is (3,2)

So, $x_{32} = \min(a_3, b_2) = \min(30,5) = 5$

So, we deleted $D_2$ Column.

**Tableau-6**

| | $D_3$ | $a_i$ |
|---|---|---|
| $S_3$ | 15 (25) | 25 |
| $b_j$ | 25 | |

In the $S_3$ Row the minimum cost cell is (3,3)

So, $x_{33} = \min(a_3, b_3) = \min(25,25) = 25$

Now, the final tableau as follows:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 7 (20) | 10 | 14 | 8 (10) | 30 |
| $S_2$ | 7 | 11 (15) | 12 | 6 (25) | 40 |
| $S_3$ | 5 | 8 (5) | 15 (25) | 9 | 30 |
| $b_j$ | 20 | 20 | 25 | 35 | |

Thus, The basic feasible solution is,

$X_{11} = 20$, $x_{14} = 10$, $x_{22} = 15$, $x_{24} = 25$, $x_{32} = 5$, $x_{33} = 25$

The cost corresponding to this feasible solution

$= 20\times7 + 10\times8 + 15\times11 + 25\times6 + 5\times8 + 25\times15$

$= 140+80+165+150+40+375$

$= 950$

Total number of variables = $m+n-1 = 3+4-1 = 6$

## 3)COLUMN-MINIMA METHOD:

In this method, we first consider the first column and find the minimum cost cell. Let, (1,1) cell be the cell in the first column with minimum cost.

**Application:** We solve the transportation problem and find the basic feasible solution using by the Column-minima method.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 1 | 5 | 8 | 6 | 8 |
| $S_2$ | 4 | 2 | 5 | 4 | 9 |
| $S_3$ | 6 | 4 | 3 | 1 | 13 |
| $b_j$ | 10 | 3 | 4 | 13 | |

**Solution:**

Here, $\sum a_i = \sum b_j = 30$

So, It is a balanced transportation problem.

**Tableau-1**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 1 (8) | 5 | 8 | 6 | 8 |
| $S_2$ | 4 | 2 | 5 | 4 | 9 |
| $S_3$ | 6 | 4 | 3 | 1 | 13 |
| $b_j$ | 10 | 3 | 4 | 13 | |

In the $D_1$ column the minimum cost cell is (1,1)

So, $x_{11} = \min(a_1, b_1) = \min(8,10) = 8$

So, we deleted $S_1$ Row.

**Tableau-2**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_2$ | 4 (2) | 2 | 5 | 4 | 9 |
| $S_3$ | 6 | 4 | 3 | 1 | 13 |
| $b_j$ | 2 | 3 | 4 | 13 | |

In the $D_1$ column the minimum cost cell is (2,1)

So, $x_{21} = \min(a_2, b_1) = \min(9,2) = 2$

So, we deleted $D_1$ Column.

**Tableau-3**

| | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|
| $S_2$ | 2 (3) | 5 | 4 | 7 |
| $S_3$ | 4 | 3 | 1 | 13 |
| $b_j$ | 3 | 4 | 13 | |

In the $D_2$ column the minimum cost cell is (2,2)

So, $x_{22} = \min(a_2, b_2) = \min(7,3) = 3$

So, we deleted $D_2$ Column.

**Tableau-4**

| | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|
| $S_2$ | 5 | 4 | 4 |
| $S_3$ | 3 (4) | 1 | 13 |
| $b_j$ | 4 | 13 | |

In the $D_3$ column the minimum cost cell is (3,3)

So, $x_{33} = \min(a_3, b_3) = \min(13,4) = 4$

So, we deleted $D_3$ Column.

**Tableau-5**

| | $D_4$ | $a_i$ |
|---|---|---|
| $S_2$ | 4 | 4 |
| $S_3$ | 1 (9) | 9 |
| $b_j$ | 13 | |

In the $D_4$ column the minimum cost cell is (3,4)

So, $x_{34} = \min(a_3, b_4) = \min(9,13) = 9$

So, we deleted $S_3$ Row.

**Tableau-6**

| | $D_4$ | $a_i$ |
|---|---|---|
| $S_2$ | 4 (4) | 4 |
| $b_j$ | 4 | |

In the $D_4$ column the minimum cost cell is (2,4)

So, $x_{24} = \min(a_2, b_4) = \min(4,4) = 4$

Now, the final tableau as follows:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 1 (8) | 5 | 8 | 6 | 8 |
| $S_2$ | 4 (2) | 2 (3) | 5 | 4 (4) | 9 |
| $S_3$ | 6 | 4 | 3 (4) | 1 (9) | 13 |
| $b_j$ | 10 | 3 | 4 | 13 | |

Thus, The basic feasible solution is,

$X_{11} = 8$, $x_{21} = 2$, $x_{22} = 3$, $x_{24} = 4$, $x_{33} = 4$, $x_{34} = 9$

The cost corresponding to this feasible solution

$= 8\times1 + 2\times4 + 3\times2 + 4\times4 + 4\times3 + 9\times1$

= 8+8+6+16+12+9

= 59

Total number of variables = m+n-1 = 3+4-1 = 6

## 4) MATRIX-MINIMA METHOD:

In this method, we first find out the cell with minimum cost in the cost matrix and allocate in that cell the maximum allowable amount. We then cross out the satisfied row or column and adjust the amounts of supply and demand accordingly. We repeat the process with the uncross ed out matrix and we are left at the end with exactly one uncrossed out row or column.

**Application:** We solve the transportation problem and find the basic feasible solution using by the Matrix-minima method.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 2 | 2 | 2 | 1 | 3 |
| $S_2$ | 10 | 8 | 5 | 4 | 7 |
| $S_3$ | 7 | 6 | 6 | 8 | 5 |
| $b_j$ | 4 | 3 | 4 | 4 | |

**Solution:**

Here, $\sum a_i = \sum b_j = 15$

So, It is a balanced transportation problem.

**Tableau-1**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 2 | 2 | 2 | 1 (3) | 3 |
| $S_2$ | 10 | 8 | 5 | 4 | 7 |
| $S_3$ | 7 | 6 | 6 | 8 | 5 |
| $b_j$ | 4 | 3 | 4 | 4 | |

In the cost matrix, cell (1,4)

So, $x_{14} = \min(a_1, b_4) = \min(3,4) = 3$

So, we deleted $S_1$ Row.

**Tableau-2**

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_2$ | 10 | 8 | 5 | 4 (1) | 7 |
| $S_3$ | 7 | 6 | 6 | 8 | 5 |
| $b_j$ | 4 | 3 | 4 | 1 | |

In the cost matrix, cell (2,4)

So, $x_{24} = \min(a_2, b_4) = \min(7,1) = 1$

So, we deleted $D_1$ Column.

**Tableau-3**

| | $D_1$ | $D_2$ | $D_3$ | $a_i$ |
|---|---|---|---|---|
| $S_2$ | 10 | 8 | 5 (4) | 6 |
| $S_3$ | 7 | 6 | 6 | 5 |
| $b_j$ | 4 | 3 | 4 | |

In the cost matrix, cell (2,3)

So, $x_{23}$ = min ($a_2$,$b_3$) = min (6,4) = 4

So, we deleted $D_3$ Column.

**Tableau-4**

| | $D_1$ | $D_2$ | $a_i$ |
|---|---|---|---|
| $S_2$ | 10 | 8 | 2 |
| $S_3$ | 7 | 6 (3) | 5 |
| $b_j$ | 4 | 3 | |

In the cost matrix, cell (3,2)

So, $x_{32}$ = min ($a_3$,$b_2$) = min (5,3) = 3

So, we deleted $D_2$ Column.

**Tableau-5**

| | $D_1$ | $a_i$ |
|---|---|---|
| $S_2$ | 10 | 2 |
| $S_3$ | 7 (2) | 2 |
| $b_j$ | 4 | |

In the cost matrix, cell (3,1)

So, $x_{31} = \min(a_3,b_1) = \min(2,4) = 2$

So, we deleted $S_3$ Row.

**Tableau-6**

| | $D_1$ | $a_i$ |
|---|---|---|
| $S_2$ | 10 (2) | 2 |
| $b_j$ | 2 | |

In the cost matrix, cell (2,1)

So, $x_{21} = \min(a_2,b_1) = \min(2,2) = 2$

Now, the final tableau as follows:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 2 | 2 | 2 | 1 (3) | 3 |
| $S_2$ | 10 (2) | 8 | 5 (4) | 4 (1) | 7 |
| $S_3$ | 7 (2) | 6 (3) | 6 | 8 | 5 |
| $b_j$ | 4 | 3 | 4 | 4 | |

Thus, The basic feasible solution is,

$x_{14} = 3$, $x_{21} = 2$, $x_{23} = 4$, $x_{24} = 1$, $x_{31} = 2$, $x_{32} = 3$

The cost corresponding to this feasible solution

$= 3\times1 + 2\times10 + 4\times5 + 1\times4 + 2\times7 + 3\times6$

$= 3+20+20+4+14+18$

$= 79$

Total number of variables = $m+n-1 = 3+4-1 = 6$

## 5) VOGEL'S APPROXIMATION METHOD (VAM):

***Step-1:***

Calculate the penalties for each row (column) by taking the difference between the smallest unit transportation cost in the same row (column). This difference indicates the penalty or extra cost that has to be paid if decision-maker fails to allocate to the cell with the minimum unit transportation cost.

*Step-2:*

select the row or column with the largest penalty and allocate as much as possible in the cell that has the least cost in the selected row or column and satisfies the rim condition. If there is a tie in the values of penalties, it can be broken by selecting the cell where the maximum allocation can be made.

*Step-3:*

Adjust the supply and demand and cross out the satisfied row or column. If a row and a column are satisfied simultaneously. Only one of them is crossed out and the remaining row (column) is assigned a zero supply (demand). Any row or column with zero supply or demand should not be used in computing future penalties.

*Step-4:*

Repeat steps 1 to 3 until the available supply at various sources and demand at various destinations is satisfied.

**Application:** We solve the transportation problem and find the basic feasible solution using by the Vogel's Approximation Method.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 5 | 3 | 6 | 4 | 30 |
| $S_2$ | 3 | 4 | 7 | 8 | 15 |
| $S_3$ | 9 | 6 | 5 | 8 | 15 |
| $b_j$ | 10 | 25 | 18 | 7 | |

**Solution:**

Here, $\sum a_i = \sum b_j = 60$

So, It is a balanced transportation problem.

An initial basic feasible solution by VAM is shown in the following table:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $S_1$ | 5 | 3 (23) | 6 | 4 (7) | 30(1) | 23(2) | 23(3) | | |
| $S_2$ | 3 (10) | 4 (2) | 7 (3) | 8 | 15(1) | 15(1) | 5(3) | 5(3) | 3 |
| $S_3$ | 9 | 6 | 5 (15) | 8 | 15(1) | 15(1) | 15(1) | 15(1) | 15 |
| $b_j$ | 10 (2) | 25 (1) | 18 (1) | 7 (4) | | | | | |
| | 10 (2) | 25 (1) | 18 (1) | | | | | | |
| | | 25 (1) | 18 (1) | | | | | | |
| | | 2 (2) | 18 (2) | | | | | | |
| | | | 18 | | | | | | |

Now, the final tableau as follows:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 5 | 3 (23) | 6 | 4 (7) | 30 |
| $S_2$ | 3 (10) | 4 (2) | 7 (3) | 8 | 15 |
| $S_3$ | 9 | 6 | 5 (15) | 8 | 15 |
| $b_j$ | 10 | 25 | 18 | 7 | |

Thus, The basic feasible solution is,

$X_{12} = 23$, $x_{14} = 7$, $x_{21} = 10$, $x_{22} = 2$, $x_{23} = 3$, $x_{33} = 15$

The cost corresponding to this feasible solution

$= 23\times3 + 7\times4 + 10\times3 + 2\times4 + 3\times7 + 15\times5$

$= 69+28+30+8+21+75$

$= 231$

Total number of variables = $m+n-1 = 3+4-1 = 6$

# Optimality Test

The test of optimality being by calculating an opportunity cost associated with each unoccupied cell in the transportation table. An unoccupied cell with the largest negative opportunity cost is selected to in the new set of transportation allocations.

We discuss here, The UV-Method (or MODI method).

## THE UV-METHOD (OR MODI METHOD):

The steps to evaluate unoccupied cells are as follows:

***Step-1:***

First find a basic feasible solution of the given transportation problem by anyone of the method discussed earlier. For an initial basic feasible solution with m+n-1 occupied cells, calculate $u_i$ and $v_j$ for rows and columns.

***Step-2:***

Determine of a set (m+n) numbers $u_i$ and $v_j$, i=1,2...m, j=1,2...n, such that for all occupied (i,j) cells $c_{ij} = u_i+v_j$. In practice to find $u_i$ and $v_j$ put any one of them equal to zero and then considering the relations $c_{ij} = u_i+v_j$ for occupied cells, all other $u_i$ and $v_j$ can be found out. Generally, that $u_i$ or $v_j$ for which the corresponding row or column contains maximum number of occupied cells is put to zero.

*Step-3:*

Calculate the cell evaluations $A_{ij}$ for each unoccupied (i,j) cells by the formula $A_{ij} = u_i + v_j - c_{ij}$ and pit them in the upper right corners of the corresponding unoccupied cells.

1) If $A_{ij} < 0$, then the solution is optimal and unique.
2) If $A_{ij} < 0$, with a least one $A_{ij} = 0$, then the solution is optimal but not unique.
3) If $A_{ij} > 0$, then the solution is not optimal.

*Step-4:*

Construct a closed path (or loop) for the unoccupied cell with the largest positive value of $A_{ij}$ start with the closed path with the selected unoccupied cell and mark a plus sign (+) in this cell. Trace a path along the rows (or columns) to an occupied cell, mark the corner with a minus sign (-) and continue down the column (or row) to an occupied cell.

Then mark the corner with plus sign (+) and minus sign (-) alternatively. Closed the path back to the selected unoccupied cell. Starting from this cell, allocate an amount θ with alternative positive and negative signs to all the ends points of the closed loop so that supply and demand constraints are always satisfied.

*Step-5:*

Select the smallest quantity amongst the cells marked with minus sign on the corners of closed loop. Allocate this value to the select unoccupied cell, add it to occupied cells marked with plus signs, and subtract it from the occupied cells marked with minus signs.

*Step-6:*

Obtain a new improved solution by allocating units to the unoccupied cell according to step 5 and calculate the new total transportation cost.

**Application-1:** We solve the transportation problem and find the optimal solution using by the UV-Method.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 2 | 2 | 2 | 1 | 3 |
| $S_2$ | 10 | 8 | 5 | 4 | 7 |
| $S_3$ | 7 | 6 | 6 | 8 | 5 |
| $b_j$ | 4 | 3 | 4 | 4 | |

**Solution:**

Here, $\sum a_i = \sum b_j = 15$

So, It is a balanced transportation problem.

An initial basic feasible solution is obtained by VAM is shown below:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_1$ | 2 ③ | 2 | 2 | 1 | 3(1) | | | | | |
| $S_2$ | 10 | 8 | 5 ③ | 4 ④ | 7(1) | 7(1) | 3(3) | | | |
| $S_3$ | 7 ① | 6 ③ | 6 ① | 8 | 5(1) | 5(1) | 5(1) | 5(1) | 4(1) | 1 |
| $b_j$ | 4 (5) | 3 (4) | 4 (3) | 4 (4) | | | | | | |
| | 1 (3) | 3 (2) | 4 (1) | 4 (4) | | | | | | |
| | 1 (3) | 3 (2) | 4 (1) | | | | | | | |
| | 1 | 3 | 1 | | | | | | | |
| | 1 | 3 | | | | | | | | |
| | 1 | | | | | | | | | |

Now, the optimal solution is obtained by using UV-Method by usual technique and is shown in the following table:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $u_i$ |
|---|---|---|---|---|---|
| $S_1$ | 2 ③ | 2 [-1] | 2 [-1] | 1 [-1] | $u_1 = -5$ |
| $S_2$ | 10 [-4] | 8 [-3] | 5 ③ | 4 ④ | $u_2 = -1$ |
| $S_3$ | 7 ① | 6 ③ | 6 ① | 8 [-3] | $u_3 = 0$ |
| $v_j$ | $v_1 = 7$ | $v_2 = 6$ | $v_3 = 6$ | $v_4 = 5$ | |

Let us assume $u_3$=0 (since 3rd row contains maximum number of basic cell)

| **Basic cell** | **$c_{ij} = u_i+v_j$** | **values of $u_i$ and $v_j$** |
|---|---|---|
| $x_{11}$ | $u_1+v_1= 2$ | $v_1 = 7$ ; $u_1 = -5$ |
| $x_{23}$ | $u_2+v_3= 5$ | $v_3 = 6$ ; $u_2 = -1$ |
| $x_{24}$ | $u_2+v_4= 4$ | $u_2 = -1$ ; $v_4 = 5$ |
| $x_{31}$ | $u_3+v_1= 7$ | $u_3 = 0$ ; $v_1 = 7$ |
| $x_{32}$ | $u_3+v_2= 6$ | $u_3 = 0$ ; $v_2 = 6$ |
| $x_{33}$ | $u_3+v_3= 6$ | $u_3 = 0$ ; $v_3 = 6$ |

| **Non-basic cell** | $\underline{A_{ij} = u_i+v_j-c_{ij}}$ |
|---|---|
| $x_{12}$ | $A_{12} = u_1+v_2-c_{12} = -5+6-2 = -1$ |
| $x_{13}$ | $A_{13} = u_1+v_3-c_{13} = -5+6-2 = -1$ |
| $x_{14}$ | $A_{14} = u_1+v_4-c_{14} = -5+5-1 = -1$ |
| $x_{21}$ | $A_{21} = u_2+v_1-c_{21} = -1+7-10 = -4$ |
| $x_{22}$ | $A_{22} = u_2+v_2-c_{22} = -1+6-8 = -3$ |
| $x_{34}$ | $A_{34} = u_3+v_4-c_{34} = 0+5-8 = -3$ |

Since, The cell evaluations of all the non-basic cells are negative.

So, Optimality has been reached.

And, the optimal solution is,

$x_{11} = 3, x_{23} = 3, x_{24} = 4, x_{31} = 1, x_{32} = 3, x_{33} = 1$

Total minimum cost = 3×2 + 3×5 + 4×4 + 1×7 + 3×6 + 1×6

= 6+15+16+7+18+6

= 68

Total number of variables = m+n-1 = 3+4-1 = 6

__Application-2:__ We solve the transportation problem and find the optimal solution using by the UV-Method.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 19 | 30 | 50 | 10 | 7 |
| $S_2$ | 70 | 30 | 40 | 60 | 9 |
| $S_3$ | 40 | 8 | 70 | 20 | 18 |
| $b_j$ | 5 | 8 | 7 | 14 | |

**Solution:**

Here, $\sum a_i = \sum b_j = 34$

So, It is a balanced transportation problem.

An initial basic feasible solution is obtained by VAM is shown below:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_1$ | 19 (5) | 30 | 50 | 10 (2) | 7(9) | 7(9) | 2(40) | 2(40) | | |
| $S_2$ | 70 | 30 | 40 (7) | 60 (2) | 9(10) | 9(20) | 9(20) | 9(20) | 9(20) | 2 |
| $S_3$ | 40 | 8 (8) | 70 | 20 (10) | 18(12) | 10(20) | 10(50) | | | |
| $b_j$ | 5 (21) | 8 (22) | 7 (10) | 14 (10) | | | | | | |
| | 5 (21) | | 7 (10) | 14 (10) | | | | | | |
| | | | 7 (10) | 14 (10) | | | | | | |
| | | | 7 (10) | 4 (50) | | | | | | |
| | | | 7 | 2 | | | | | | |
| | | | | 2 | | | | | | |

Now, the optimal solution is obtained by using UV-Method by usual technique and is shown in the following table:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $u_i$ |
|---|---|---|---|---|---|
| $S_1$ | 19 (5) | 30 [-32] | 50 [-60] | 10 (2) | $u_1 = 10$ |
| $S_2$ | 70 [-1] | 30 +θ [18] | 40 (7) | 60 -θ (2) | $u_2 = 60$ |
| $S_3$ | 40 [-11] | 8 -θ (8) | 70 [-70] | 20 +θ (10) | $u_3 = 20$ |
| $v_j$ | $v_1 = 9$ | $v_2 = -12$ | $v_3 = -20$ | $v_4 = 0$ | |

Let us assume $v_4 = 0$ (since, 4th column contains maximum number of the basic cell)

| **Basic cell** | **$c_{ij} = u_i+v_j$** | **values of $u_i$ and $v_j$** |
|---|---|---|
| $x_{11}$ | $u_1+v_1= 19$ | $u_1 = 10$ ; $v_1 = 9$ |
| $x_{14}$ | $u_1+v_4= 10$ | $v_4 = 0$ ; $u_1 = 10$ |
| $x_{23}$ | $u_2+v_3= 40$ | $u_2 = 60$ ; $v_3 = -20$ |
| $x_{24}$ | $u_2+v_4= 60$ | $v_4 = 0$ ; $u_2 = 60$ |
| $x_{32}$ | $u_3+v_2= 8$ | $u_3 = 20$ ; $v_2 = -12$ |
| $x_{34}$ | $u_3+v_4= 20$ | $v_4 = 0$ ; $u_3 = 20$ |

| **Non-basic cell** | **$A_{ij} = u_i+v_j-c_{ij}$** |
|---|---|
| $x_{12}$ | $A_{12} = u_1+v_2-c_{12} = 10-12-30 = -32$ |
| $x_{13}$ | $A_{13} = u_1+v_3-c_{13} = 10-20-50 = -60$ |

| | |
|---|---|
| $x_{21}$ | $A_{21} = u_2+v_1-c_{21} = 60+9-70 = -1$ |
| $x_{22}$ | $A_{22} = u_2+v_2-c_{22} = 60-12-30 = 18$ |
| $x_{31}$ | $A_{31} = u_3+v_1-c_{31} = 20+9-40 = -11$ |
| $x_{33}$ | $A_{33} = u_3+v_3-c_{33} = 20-20-70 = -70$ |

Since, $A_{22}$ is positive (>0), so this solution is not optimal.

Now, we from a loop with the cell (2,2) and the basic cells (2,4),(3,4) and (3,2) as shown in the previous table.

Then, we put θ with alternate signs to this four cells forming the loop as shown in the same table.

Now, we are to choose that minimum value of θ = min (2,8) = 2

Now, the new basic feasible solution is given in the next table:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $u_i$ |
|---|---|---|---|---|---|
| $S_1$ | 19 (5) | 30 [-32] | 50 [-42] | 10 (2) | $u_1 = -32$ |
| $S_2$ | 70 [-19] | 30 (2) | 40 (7) | 60 [-18] | $u_2 = 0$ |
| $S_3$ | 40 [-11] | 8 (6) | 70 [-52] | 20 (12) | $u_3 = -22$ |
| $v_j$ | $v_1 = 51$ | $v_2 = 30$ | $v_3 = 40$ | $v_4 = 42$ | |

Let us assume $u_2 = 0$

| Basic cell | $c_{ij} = u_i+v_j$ | values of $u_i$ and $v_j$ |
|---|---|---|
| $x_{11}$ | $u_1+v_1= 19$ | $u_1 = -32$ ; $v_1 = 51$ |
| $x_{14}$ | $u_1+v_4= 10$ | $v_4 = 42$ ; $u_1 = -32$ |
| $x_{22}$ | $u_2+v_2= 30$ | $u_2 = 0$ ; $v_2 = 30$ |
| $x_{23}$ | $u_2+v_3= 40$ | $u_2 = 0$ ; $v_3 = 40$ |
| $x_{32}$ | $u_3+v_2= 8$ | $v_2 = 30$ ; $u_3 = -22$ |
| $x_{34}$ | $u_3+v_4= 20$ | $u_3 = -22$ ; $v_4 = 42$ |

| Non-basic cell | $A_{ij} = u_i+v_j-c_{ij}$ |
|---|---|
| $x_{12}$ | $A_{12} = u_1+v_2-c_{12} = -32+30-30 = -32$ |
| $x_{13}$ | $A_{13} = u_1+v_3-c_{13} = -32+40-50 = -42$ |
| $x_{21}$ | $A_{21} = u_2+v_1-c_{21} = 0+51-70 = -19$ |
| $x_{24}$ | $A_{24} = u_2+v_4-c_{24} = 0+42-60 = -18$ |
| $x_{31}$ | $A_{31} = u_3+v_1-c_{31} = -22+51-40 = -11$ |
| $x_{33}$ | $A_{33} = u_3+v_3-c_{33} = -22+40-70 = -52$ |

Since, the cell evaluations of all the non basic cells are negative.

So, Optimality has been reached.

And, the optimal solution is,

$x_{11} = 5$, $x_{14} = 2$, $x_{22} = 2$, $x_{23} = 7$, $x_{32} = 6$, $x_{34} = 12$

Total minimum cost = 5×19 + 2×10 + 2×30 + 7×40 + 6×8 + 12×20

= 95+20+60+280+48+240

= 743

Total number of variables = m+n-1 = 3+4-1 = 6

# Unbalanced Transportation problem

We know that for existence of a feasible solution to a transportation problem it is necessary that total supply must be equal to total demand, that is

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

But when,

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

i.e. total supply is not equal to demand, then the transportation is called unbalanced transportation problem.

To solve any unbalanced transportation problem, we first convert this problem to a balanced transportation problem. For this we introduce a dummy source or destination with that amount of supply or demand respectively which will be necessary to make this problem a balanced transportation problem. The transportation cost from any dummy source or to a dummy destination is taken to be zero.

Dummy destination will be assumed to be $\sum a_i - \sum b_j$ (since, $\sum a_i > \sum b_j$)

and, Dummy source will be assumed to be $\sum b_j - \sum a_i$ (since, $\sum b_j > \sum a_i$)

Application-1: We solve the following transportation problem.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 6 | 1 | 9 | 3 | 70 |
| $S_2$ | 11 | 5 | 2 | 8 | 55 |
| $S_3$ | 10 | 12 | 4 | 7 | 70 |
| $b_j$ | 85 | 35 | 50 | 45 | |

**Solution:**

Here, $\sum a_i = 195$ and $\sum b_j = 215$

So, It is a unbalanced transportation problem.

Since, $\sum b_j > \sum a_i$

So, with think of a dummy source with supply $\sum b_j - \sum a_i = 215 - 195 = 20$ with zero transportation constant.

So, the corresponding balanced transportation problem is shown in the following table:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 6 | 1 | 9 | 3 | 70 |
| $S_2$ | 11 | 5 | 2 | 8 | 55 |
| $S_3$ | 10 | 12 | 4 | 7 | 70 |
| $S_4$ | 0 | 0 | 0 | 0 | 20 |
| $b_j$ | 85 | 38 | 50 | 45 | |

Now, we find the initial basic feasible solution.

We use VAM with usual process as shown in the next table.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_1$ | 6 (65) | 1 (5) | 9 | 3 | 70(2) | 70(2) | 5(2) | | | |
| $S_2$ | 11 | 5 (30) | 2 (25) | 8 | 55(3) | 55(3) | 55(3) | 55(3) | 25(6) | |
| $S_3$ | 10 | 12 | 4 (25) | 7 (45) | 70(3) | 70(3) | 70(3) | 70(3) | 70(3) | 70 |
| $S_4$ | 0 (20) | 0 | 0 | 0 | 20(0) | | | | | |
| $b_j$ | 85 (6) | 35 (1) | 50 (2) | 45 (3) | | | | | | |
| | 65 (4) | 35 (4) | 50 (2) | 45 (4) | | | | | | |
| | | 35 (4) | 50 (2) | 45 (4) | | | | | | |
| | | 30 (7) | 50 (2) | 45 (1) | | | | | | |
| | | | 50 (2) | 45 (1) | | | | | | |
| | | | 25 | 45 | | | | | | |

Number of basic cells = m+n-1 = 4+4-1 = 7

So, The problem is non-degenerate.

Now, We find the extreme solution,

we use U-V method as usual and the scheme is shown in following table:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $u_i$ |
|---|---|---|---|---|---|
| $S_1$ | 6, -θ, (65) | 1, +θ, (5) | 9, [-11] | 3, [-2] | $u_1 = 0$ |
| $S_2$ | 11, [-1] | 5, -θ, (30) | 2, +θ, (25) | 8, [-3] | $u_2 = 4$ |
| $S_3$ | 10, +θ, [2] | 12, [-5] | 4, -θ, (25) | 7, (45) | $u_3 = 6$ |
| $S_4$ | 0, (20) | 0, [-5] | 0, [-8] | 0, [-5] | $u_4 = -6$ |
| $v_j$ | $v_1 = 6$ | $v_2 = 1$ | $v_3 = -2$ | $v_4 = 1$ | |

Let us assume $u_1 = 0$

| **Basic cell** | **$c_{ij} = u_i+v_j$** | **values of $u_i$ and $v_j$** |
|---|---|---|
| $x_{11}$ | $u_1+v_1= 6$ | $u_1 = 0$ ; $v_1 = 6$ |
| $x_{12}$ | $u_1+v_2= 1$ | $u_1 = 0$ ; $v_2 = 1$ |
| $x_{22}$ | $u_2+v_2= 5$ | $v_2 = 1$ ; $u_2 = 4$ |
| $x_{23}$ | $u_2+v_3= 2$ | $u_2 = 4$ ; $v_3 = -2$ |
| $x_{33}$ | $u_3+v_3= 4$ | $v_3 = -2$ ; $u_3 = 6$ |
| $x_{34}$ | $u_3+v_4= 7$ | $u_3 = 6$ ; $v_4 = 1$ |
| $x_{41}$ | $u_4+v_1= 0$ | $v_1 = 6$ ; $u_4 = -6$ |

| **Non-basic cell** | **$A_{ij} = u_i+v_j-c_{ij}$** |
|---|---|
| $x_{13}$ | $A_{13} = u_1+v_3-c_{13} = 0-2-9 = -11$ |
| $x_{14}$ | $A_{14} = u_1+v_4-c_{14} = 0+1-3 = -2$ |
| $x_{21}$ | $A_{21} = u_2+v_1-c_{21} = 4+6-11 = -1$ |

| | |
|---|---|
| $x_{24}$ | $A_{24} = u_2+v_4-c_{24} = 4+1-8 = -3$ |
| $x_{31}$ | $A_{31} = u_3+v_1-c_{31} = 6+6-10 = 2$ |
| $x_{32}$ | $A_{32} = u_3+v_2-c_{32} = 6+1-12 = -5$ |
| $x_{42}$ | $A_{42} = u_4+v_2-c_{42} = -6+1-0 = -5$ |
| $x_{43}$ | $A_{43} = u_4+v_3-c_{43} = -6-2-0 = -8$ |
| $x_{44}$ | $A_{44} = u_4+v_4-c_{44} = -6+1-0 = -5$ |

We see that, $A_{31}$ is positive.

So, this solution is not optimal.

Now, we form a loop with the cell (3,1) and the basic cells (3,3), (2,3), (2,2), (1,2) and (1,1) as shown in the previous table.

Then we put θ alternate signs to this six cells forming the loop as shown in the same table.

Now, we put, minimum value of θ = (65,30,25) = 25

Now, the new basic feasible solution is given in the next table:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $u_i$ |
|---|---|---|---|---|---|
| $S_1$ | 6 (40) | 1 (30) | 9 [-11] | 3 [0] | $u_1 = 6$ |
| $S_2$ | 11 [-1] | 5 (5) | 2 (50) | 8 [-1] | $u_2 = 10$ |
| $S_3$ | 10 (25) | 12 [-7] | 4 [-2] | 7 (45) | $u_3 = 10$ |
| $S_4$ | 0 (20) | 0 [-5] | 0 [-8] | 0 [-3] | $u_4 = 0$ |
| $v_j$ | $v_1 = 0$ | $v_2 = -5$ | $v_3 = -8$ | $v_4 = -3$ | |

Let us assume $v_1 = 0$

| Basic cell | $c_{ij} = u_i+v_j$ | values of $u_i$ and $v_j$ |
|---|---|---|
| $x_{11}$ | $u_1+v_1= 6$ | $v_1 = 0$ ; $u_1 = 6$ |
| $x_{12}$ | $u_1+v_2= 1$ | $u_1 = 6$ ; $v_2 = -5$ |
| $x_{22}$ | $u_2+v_2= 5$ | $v_2 = -5$ ; $u_2 = 10$ |
| $x_{23}$ | $u_2+v_3= 2$ | $u_2 = 10$ ; $v_3 = -8$ |
| $x_{31}$ | $u_3+v_1= 10$ | $v_1 = 0$ ; $u_3 = 10$ |
| $x_{34}$ | $u_3+v_4= 7$ | $u_3 = 10$ ; $v_4 = -3$ |
| $x_{41}$ | $u_4+v_1= 0$ | $v_1 = 0$ ; $u_4 = 0$ |

| Non-basic cell | $\Delta_{ij} = u_i+v_j-c_{ij}$ |
|---|---|
| $x_{13}$ | $\Delta_{13} = u_1+v_3-c_{13} = 6-8-9 = -11$ |
| $x_{14}$ | $\Delta_{14} = u_1+v_4-c_{14} = 6-3-3 = 0$ |
| $x_{21}$ | $\Delta_{21} = u_2+v_1-c_{21} = 10+0-11 = -1$ |
| $x_{24}$ | $\Delta_{24} = u_2+v_4-c_{24} = 10-3-8 = -1$ |
| $x_{32}$ | $\Delta_{32} = u_3+v_2-c_{32} = 10-5-12 = -7$ |
| $x_{33}$ | $\Delta_{33} = u_3+v_3-c_{33} = 10-8-4 = -2$ |
| $x_{42}$ | $\Delta_{42} = u_4+v_2-c_{42} = 0-5-0 = -5$ |
| $x_{43}$ | $\Delta_{43} = u_4+v_3-c_{43} = 0-8-0 = -8$ |
| $x_{44}$ | $\Delta_{44} = u_4+v_4-c_{44} = 0-3-0 = -3$ |

Since, the cell evaluations of all non-basic cells are negative or zero.

So, Optimality has been reached.

The optimal solution is,

$x_{11} = 40$, $x_{12} = 30$, $x_{22} = 5$, $x_{23} = 50$, $x_{31} = 25$, $x_{34} = 45$

The minimum cost = 40×6 + 30×1 + 5×5 + 50×2 + 25×10 + 45×7

= 240+30+25+100+250+315 = 960

**Application-2:** We solve the following transportation problem.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $a_i$ |
|---|---|---|---|---|---|
| $S_1$ | 42 | 48 | 38 | 37 | 160 |
| $S_2$ | 40 | 49 | 52 | 51 | 150 |
| $S_3$ | 39 | 38 | 40 | 43 | 190 |
| $b_j$ | 80 | 90 | 110 | 160 | |

**Solution:**

Here, $\sum a_i = 500$ and $\sum b_j = 440$

So, It is a unbalanced transportation problem.

Since, $\sum a_i > \sum b_j$

So, with think of a dummy destination with demand $\sum a_i - \sum b_j = 500 - 440 = 60$ with zero transportation constant.

So, the corresponding balanced transportation problem is shown in the following table:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $a_i$ |
|---|---|---|---|---|---|---|
| $S_1$ | 42 | 48 | 38 | 37 | 0 | 160 |
| $S_2$ | 40 | 49 | 52 | 51 | 0 | 150 |
| $S_3$ | 39 | 38 | 40 | 43 | 0 | 190 |
| $b_j$ | 80 | 90 | 110 | 160 | 60 | |

Now, we find the initial basic feasible solution.

We use VAM with usual process as shown in the next table.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $a_i$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $S_1$ | 42 | 48 | 38 | 37 (160) | 0 | 160(1) | 160(1) | 160(1) | | | |
| $S_2$ | 40 (80) | 49 | 52 (10) | 51 | 0 | 150(9) | 150(11) | 70(1) | 70 | 70 | 60 |
| $S_3$ | 39 | 38 (90) | 40 (100) | 43 | 0 | 190(1) | 100(1) | 100(3) | 100 | | |
| $b_j$ | 80 (1) | 90 (10) | 110 (2) | 160 (6) | 60 | | | | | | |
| | 80 (1) | | 110 (2) | 160 (6) | 60 | | | | | | |
| | | | 110 (2) | 160 (6) | 60 | | | | | | |
| | | | 110 (2) | | 60 | | | | | | |
| | | | 10 | | 60 | | | | | | |
| | | | | | 60 | | | | | | |

Number of basic cells = 5 (<m+n-1 = 3+4-1 = 6)
So, it is a degenerate basic solution.
Let us introduce an ε a very small positive quantity, at the empty cell (1,3) which is least cost empty cell and proceed for optimal solution.

Now, we use U-V method as usual and the scheme is shown in following table:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $u_i$ |
|---|---|---|---|---|---|
| $S_1$ | 42 [-16] | 4 [-12] | 38 (ε) | 37 (160) | $u_1 = 0$ |
| $S_2$ | 40 (80) | 49 +θ [1] | 52 -θ (10) | 51 [0] | $u_2 = 14$ |
| $S_3$ | 39 [-11] | 38 -θ (90) | 40 +θ (100) | 43 [-4] | $u_3 = 2$ |
| $v_j$ | $v_1 = 26$ | $v_2 = 36$ | $v_3 = 38$ | $v_4 = 37$ | |

Let us assume $u_1 = 0$

| **Basic cell** | $c_{ij} = u_i+v_j$ | **values of $u_i$ and $v_j$** |
|---|---|---|
| $x_{13}$ | $u_1+v_3= 38$ | $u_1 = 0$ ; $v_3 = 38$ |
| $x_{14}$ | $u_1+v_4= 37$ | $u_1 = 0$ ; $v_4 = 37$ |
| $x_{21}$ | $u_2+v_1= 40$ | $u_2 = 14$ ; $v_1 = 26$ |
| $x_{23}$ | $u_2+v_3= 52$ | $v_3 = 38$ ; $u_2 = 14$ |
| $x_{32}$ | $u_3+v_2= 38$ | $u_3 = 2$ ; $v_2 = 36$ |

$x_{33}$ $u_3+v_3= 40$ $v_3 = 38$ ; $u_3 = 2$

**Non-basic cell** **$A_{ij} = u_i+v_j-c_{ij}$**

| Non-basic cell | $A_{ij} = u_i+v_j-c_{ij}$ |
|---|---|
| $x_{11}$ | $A_{11} = u_1+v_1-c_{11} = 0+26-42 = -16$ |
| $x_{12}$ | $A_{12} = u_1+v_2-c_{12} = 0+36-48 = -12$ |
| $x_{22}$ | $A_{22} = u_2+v_2-c_{22} = 14+36-49 = 1$ |
| $x_{24}$ | $A_{24} = u_2+v_4-c_{24} = 14+37-51 = 0$ |
| $x_{31}$ | $A_{31} = u_3+v_1-c_{31} = 2+26-39 = -11$ |
| $x_{34}$ | $A_{34} = u_3+v_4-c_{34} = 2+37-43 = -4$ |

We see that, $A_{22}$ is positive.
So, this solution is not optimal.
Now, we form a loop with the cell (2,2) and the basic cells (2,3), (3,3), (3,2) as shown in the previous table.
Then we put θ alternate signs to this four cells forming the loop as shown in the same table.
Now, we put, minimum value of θ = (90,10) = 10

Now, the new basic feasible solution is given in the next table:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $u_i$ |
|---|---|---|---|---|---|
| $S_1$ | 42 [-15] | 48 [-12] | 38 (ε) | 37 (160) | $u_1 = 0$ |
| $S_2$ | 40 (80) | 49 (10) | 52 [-1] | 51 [-1] | $u_2 = 13$ |
| $S_3$ | 39 [-10] | 38 (80) | 40 (110) | 43 [-4] | $u_3 = 2$ |
| $b_j$ | $v_1 = 27$ | $v_2 = 36$ | $v_3 = 38$ | $v_4 = 37$ | |

Let us assume $u_1 = 0$

| Basic cell | $c_{ij} = u_i+v_j$ | values of $u_i$ and $v_j$ |
|---|---|---|
| $x_{13}$ | $u_1+v_3= 38$ | $u_1 = 0$ ; $v_3 = 38$ |
| $x_{14}$ | $u_1+v_4= 37$ | $u_1 = 0$ ; $v_4 = 37$ |
| $x_{21}$ | $u_2+v_1= 40$ | $u_2 = 13$ ; $v_1 = 27$ |
| $x_{22}$ | $u_2+v_2= 49$ | $v_2 = 36$ ; $u_2 = 13$ |
| $x_{32}$ | $u_3+v_2= 38$ | $u_3 = 2$ ; $v_2 = 36$ |
| $x_{33}$ | $u_3+v_3= 40$ | $v_3 = 38$ ; $u_3 = 2$ |

| Non-basic cell | $A_{ij} = u_i+v_j-c_{ij}$ |
|---|---|
| $x_{11}$ | $A_{11} = u_1+v_1-c_{11} = 0+27-42 = -15$ |
| $x_{12}$ | $A_{12} = u_1+v_2-c_{12} = 0+36-48 = -12$ |
| $x_{23}$ | $A_{23} = u_2+v_3-c_{23} = 13+38-52 = -1$ |
| $x_{24}$ | $A_{24} = u_2+v_4-c_{24} = 13+37-51 = -1$ |
| $x_{31}$ | $A_{31} = u_3+v_1-c_{31} = 2+27-39 = -10$ |
| $x_{34}$ | $A_{34} = u_3+v_4-c_{34} = 2+37-43 = -4$ |

Since, the cell evaluations of all non-basic cells are negative.
So, Optimality has been reached.

The optimal solution is,

$x_{14} = 160$, $x_{21} = 80$, $x_{22} = 10$, $x_{32} = 80$, $x_{33} = 110$

The minimum cost = 160×37 + 80×40 + 10×49 + 80×38 + 110×40

= 5920+3200+490+3040+4400

= 17050

# CONCLUSION

The transportation cost is a significant component of the total cost structure for any business the transportation problem was formulated as a Linear Programming and illuminated with the standard LP solvers, for example, the Management researcher module to get the ideal arrangement. The computational outcomes gave the insignificant total transportation cost and the values for the decision factors for optimality.

After illuminating the LP (linear programming) problems by the PC bundle, the optimum arrangements gave the important information, for example, affectability examination to settle on ideal decisions. Using this scientific model (Transportation Model) the business can identify effectively and proficiently plan out its transportation, with the goal that it cannot just limit the cost of shipping goods and administrations yet in addition make time utility by arriving at the goods promotion administrations at the ideal spot advertisement ideal time. This means will empower them to meet the corporative objective, for example, instruction reserve, amusement and other help they offered to individuals.

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# ABHEDANANDA MAHAVIDYALA

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SEMESTER - VI

DEPARTMENT OF MATHEMATICS


A project work presented for the degree of Bachelor of Science.

TOPIC- Ring and it's properties

# DEPARTMENT OF MATHEMATICS

**ABHEDANANDA MAHAVIDYALAYA**

SAINTHIA, BIRBHUM

**CERTIFICATE**

This is to certify that **PRANOBENDU ADHIKARI** of semester VI bearing Roll No 210330100028 has successfully completed his/her Project on (Title) RING AND IT'S PROPERTIES under **DR. PARTHA GHOSH** in the Department of **MATHEMATICS** during the academic year 2023-24.

26.07.24

Supervisor

26/7/24

Head

Department of Mathematics

# DECLARATION

*The project topic assigned to me has been submitted. I have done it myself. It is from my own labour and free from any sort of imitation. I also declared that the given task was not submitted previously by other in the Mathematics department of Abhedananda Mahavidyalaya.*

Pranobendu Adhikari

# ACKNOWLEDGEMENT

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# INDEX

# INTRODUCTION

In mathematics ring is an algebraic structure consisting of a set together with two binary operations usually called addition and multiplication, where the set is an abelian group under addition (called the additive group of the ring) and a monoid under multiplication such that multiplication distributes over addition. In other words the ring axioms require that addition is commutative, addition and multiplication is associative, multiplication distributes over addition each element in the set has an additive inverse, and there exists an additive identity. One of the most common examples of a ring is the set of integers endowed with its natural operations of addition and multiplication.

The branch of mathematics that studies rings is known as ring theory. Ring theorists study properties common to both similar mathematical structures such as integers and polynomials, and to the many less well-known mathematical structures that also satisfy the axioms of ring theory. The ubiquity of rings makes them a central organizing principle of contemporary mathematics.

# HISTORY

The study of rings originated from the theory of polynomial rings and the theory of algebraic integers. Furthermore, the appearance of hypercomplex numbers in the mid-19th century undercut the pre-eminence of fields in mathematical analysis

In the 1880s *Richard Dedekind* introduced the concept of a ring, and the term ring (Zahlring) was coined by David Hilbert in 1892 and published in the article Die Theorie der algebraischen Zahlkörper, Jahresbericht der Deutschen Mathematiker Vereinigung, Vol. 4, 1897.

Richard Dedekind

According to Harvey Cohn, Hilbert used the term for a specific ring that had the property of "circling directly back" to an element of itself. The first axiomatic definition of a ring was given by Adolf Fraenkel in an essay in Journal für die reine und angewandte Mathematik (A. L. Crelle), vol. 145, 1914. In 1921, Emmy Noether gave the first axiomatic foundation of the theory of commutative rings in her monumental paper Ideal Theory in Rings.

# IMPORTANCE OF RING

Ring Theory is an extension of Group Theory, vibrant, wide areas of current research in mathematics, computer science and mathematical/theoretical physics. They have many applications to the study of geometric objects, to topology and in many cases their links to other branches of algebra Also ring theory may be used to understand fundamental physical laws, such as those underlying special relativity and symmetry phenomena in molecular chemistry.

# DEFINITION OF RING

A ring is a set R equipped with two binary operations +: R R→Rand RRR (where denotes the Cartesian product), called addition and multiplication. To qualify as a ring, the set and two operations, ( R_{1} +,*) must satisfy the following condition.

1. ( R ,+) is required to be an abelian group under addition :

   a) Closure under addition: For all a, b in R, the result of the operation a + b is also in R

   b) Under addition: For all a, b, c in R, the equation (a + b) + c = a + (b + c) hold.

   c) Existence of additive identity: There exists an element 0 in R, such that for all elements a in R, the equation 0 + a = a + 0 = a holds.

   d) Existence of additive inverse: For each a in R, there exists an element b in R such that a + b = b + a = 0

   e) Commutative of addition: For all a, b in R, the equation a + b = b + a holds.

2. ( R ,*) is required to be a semi group under multiplication :

   a) Closure under multiplication: For all a, b in R, the result of the operation ab is also in R.

b) Associativity of multiplication: For all a, b, c in R, the equation (ab) * c = a(bc) holds.

3. The distributive laws:

   i. For all a, b and c in R, the equation a(b + c) =(a* b)+(a*c) holds.

   ii. For all a, b and c in R, the equation (a + b) * c = (ac) + (bc) holds.

## TYPE OF RING

1. Commutative ring : A ring in which a.b = b.a for all a,b∈R is called commutative ring.

Examples:

I. ( Z ,+, bullet) is a commutative ring.

II. ( IR ,+ bullet) is a commutative ring.

III. ( Q ,+, bullet) is a commutative ring.

IV. M(IR) is not commutative ring with respect to matrix addition and multiplication:

2. Ring with unity : If in a ring, there exists an element denoted by I such that, I . a = a =a. I for all a in R, then R is called ring with unit element.

The element I \in R is called the unit element of the ring.

Examples:

i. (Z,+, bullet) is a ring with unity.

ii. $M^2$(IR) is a ring with unity

3. Null ring or zero Ring : The set R consisting of a single element 0 with two binary operations denoted by 0 + 0 = 0 and 0.0 = 0 is a ring and is called null ring.

## SOME EXAMPLE OF RING

1. (Z ,+) is a commutative group and (Z , .) is a commutative me 1 being the identity element.The distributive law holds. Therefore ( Z ,+..) is a commutative ring with unity.

- (Q.+..) is a commutative ring with unity.
- ( R .+..) is a commutative ring with unity.
- (C. +..) is a commutative ring with unity.

2. ( 2Z ,+) is a commutative group and ( 2Z ,.) is a commutative semi group. The distributive law holds.

Therefore (2Z, +..) is a commutative ring. It is a ring without unity.

3. Ring of real matrices : $M^2$(IR) be the set of all 2×2 matric whose elements are real numbers.

( $M_2$R) ,+) is a commutative group, where + denotes matrix addition and ($M_2$ (R) , ) is a monoid, where '.' denotes matrix multiplication. T distributive laws hold.

Therefore ($M_2$ (R) ,+,.) is a ring with unity. The identity matrix I is the unity in the ring. This is a non-commutative ring. Let n∈N. Then ($M_2$ (R) ,+,.) is the ring of all n x n real matrices. It is a non-commutative ring with unity I, In being the unity in the ring.

4. Ring of integers modulo n : For a fixed ne N, let Zn be the classes of residues of integers modulo n. Zn =\ 0,1 , 2 ,....,n-1\

( Zn ,+) is a commutative group, where '+' denotes addition (mod n).

( Zn ,*) is a commutative monoid where '.' denotes multiplication (mod n). The distributive law holds.

Therefore ( Zn ,+,.) is a commutative ring with unity. 1 is the unity.

5. Ring of Gaussian integers : Let us consider the subset if C given by Z[i] = (a + ib ; a, b ∈Z)

Z[i] is the set of all complex numbers of the form a + ib , where a and b are integers.

Z[i] forms a ring under addition and multiplication of complex number integers. This is a commutative ring with unity.

This ring is called the ring of Gaussian integers.

6. Ring of Gaussian numbers : Let us consider the subset of C given by Q[i] = (a + ib ; a, b ∈ Q).

Q[i] is the set of all complex numbers of the form a + ib where a and b are rational numbers.

Q[i] forms a numbers er addition and multiplication of complex numbers. This is a commutative ring with unity.

This ring is called the ring of Gaussian numbers

7. Ring of Quaternions : Let us consider the set H of 2x2 complex matrices given by

H= $\left\{\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} ;\ a, b, c, d \in \mathrm{IR}\right\}$

$\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$ can be expressed as $aI$ + bJ + cK + dL,
where $\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, L = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}$.

(H,+,.) is a ring with respect to matrix addition and matrix multiplication. This is non-commutative ring with unity, I being the unity.

This ring is called the ring of real quaternions.

8. Ring of continuous function :

Let S be the set of all real valued continuous functions on the closed and bounded interval [a,b]. Let f :[a, b] → R . g :[a, b] → R be the elements of S.

We define addition and multiplication of f and g by (f + g)(x) = f(x) + g(x) ,x ∈[a,b] (f.g)(x)=f(x).g(x) . x ∈[a, b]

( S ,+,*) is a commutative ring with unity. The function i defined by i(x) = 1 for all x in[a, b] is the unity in the ring. The function o defined by o(x) = 0 for all x in[a, b] is the zero element in the ring. This ring is denoted by C[a,b] .

9. Zero ring (Trivial Ring) :

Let ( A ,+) be an abelian group with the identity element 0. Let multiplication (.) be defined on A by b = 0 for every pair of elements a,b in A Then A is closed under multiplication.

Let a,b,c in A. Then a (b.c)= a * 0 = 0 , by definition. Also (a.b).c=0. c = 0 , by definition.

Hence multiplication is associative on A Let a,b,c on A Then a. (b + c) = 0 and a.b +a. c = 0 + 0 = 0 Thus a.(b + c)=a.b+a.c Similarly, (b + c).a=b.a+c.a Hence distributive laws hold in A.

Therefore ( A ,+,.) is a ring. This ring is called a zero-ring. Thus every abelian group is the additive group of a certain zero- ring. In particular, the element 0 in the abelian group A forms a ring by itself. This ring is called the trivial ring. In this ring 0 is the additive as well as the multiplicative identity.

10. Non trivial ring : If a ring contain at lest two element then the ring is non trivial ring.

## DIVISOR OF ZERO :

In a ring (R,+,*); a,b are said to be divisor of zero if

$$a \neq 0,\ b \neq 0 \text{ but } a*b = 0$$

then, a is left divisor of zero

and, b is right divisor of zero.

Example—

i. (Z,+,*), (R,+,*), (Q,+,*), (C,+,*) contain no divisor of zero.

ii. The ring ($Z_6$, +, *); 2, 3, 4 are divisor of zero.

Result—

i. If 'a' is unit in a ring R with unity then 'a' is not divisor of zero.

## CHARACTERSTIC OF RING :

Let (R,+,*) is a ring, n∈ $N$ is call char of ring R if na = 0 , ∀$a$∈ $R$, n is least positive integer.

If no such n exist s.t na = 0, ∀n∈ R then this ring is called zero-char.

Example—

1. Char(R) = 0 2. char(Z) = 0 3. char(Q) = 0

4. char($Z_6$) = ~~0~~ 6 5. char($Z_n$) = n

- IMDEMPOTENT –

In a ring (R,+,*), 'a' is called imdempotent element if a.a = a.

- NILPOTENT –

An element 'a' in a ring R od called Nilpotent element of index k, if k is least positive integer s.t a^k = 0.

- BOOLIAN RING –

If in a ring R , every element is a^2 = a, then if is called Boolian ring.

Examaple- $Z_2 \times Z_2$ is a Boolian ring.

INTEGRAL DOMAIN :

A non-trivial commutative ring with unity is called an Integral Domain if it contain no divisor of zero.

Example –

1. (Z,+,*), (R,+,*), (Q,+,*) are Integral Domain.
2. ($Z_p$, +, *) is an Integral Domain. Where p is prime.

Note–

The char of Integral Domain is either zero or prime.

## SKEW FIELD/ DIVISON RING :

A non-trivial ring with unity is called a skew field or division ring if every non-zero element of it has multiplicative inverse.

Example–

($Z_p$, +, *) is a skew field, Where p is prime.

Note–

A skew field contain no divisor of zero.

## FIELD :

A commutative skew field is field.

Example –

($Z_p$, +, *) is a field, Where p is prime.

Note–

Every finite Integral Domain is a field. Each finiteness is necessary.

### MIND MAP

1. (R,+) Is commutative group
2. (R,*) is semi group
3. Distributive laws

} – **RING**

**RING** + ( unity + units ) = **SKEW FIELD**

**SKEW FIELD** + (commutative laws in multiplication) = **FIELD**

| EXAMPLE | RING | WITH UNITY | UNITS | COMMUTATIVE RING | SKEW FIELD | FIELD |
|---|---|---|---|---|---|---|
| (Z,+,*) | ✓ | ✓ | ✗ | ✓ | ✗ | ✗ |
| (Q,+,*) | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| (R,+,*), | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| (2Z,+,*) | ✓ | ✗ | ✗ | ✓ | ✗ | ✗ |
| $M^2$(IR) | ✓ | ✓ | ✗ | ✗ | ✗ | ✗ |
| ($Z_4$, +, *) | ✓ | ✓ | ✗ | ✓ | ✗ | ✗ |
| (Zp, +, *) | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Polynomial ring IR[x] | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Ring of continuous function C[a,b] | ✓ | ✓ | May or May Not | ✓ | May or May Not | May or May Not |
| Z[i] = (a + ib ; a, b ∈Z) Ring of gaussian number | ✓ | ✓ | ✗ | ✓ | ✗ | ✗ |
| Q[i] = (a + ib ; a, b ∈ Q)Ring of gaussian number | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |

## SUB-RING :

A non empty subset S of a ring R is a subring of R if S itself a ring is induce operation in R.

**or**

A non empty subset S of a ring R is called a subring of (R,+,*), if

i. (S,+,*) is a ring.

ii. $S \subseteq R$

Example—

1. The subring {0} is called trivial subring of R
2. (mZ,+,*) is subring of (Z,+,*)
3. Z x Z is a ring,

   S ={ (x,0) : x∈ Z} then S is a subring of Z x Z

   T ={(x,x) ; x ∈ Z} then subring of (Q,+,*)

4. (Q,+,*) is a ring with unity (Z,+,*) is a subring of (Q,+,*)

Note—

The necessary and sufficient condition that a non empty subset S will subring of R , iff

i. $a,b \in S \Rightarrow a\text{-}b \in S$

ii. $a,b \in S \Rightarrow a.b \in S$

## SUB-FIELD :

A non empty subset K of a field F is said to be a sub-field of F if (F,+ ,*) is a field and K is sub-field of (F,+,*).

**or**

A non empty subset K of a field (F,+,*) will be sub-field of (F,+,*) if

i. $a,b \in K \Rightarrow a\text{-}b \in K$

ii. $a,b \in K$, $b \neq 0 \Rightarrow a.\, b^{-1} \in K$

## IDEAL OF RING :

Let (R,+,*) be ring S be non empty subset of R then S is a ideal of R iff

i. $a - b \in S, \forall a,b \in S$

ii. $a \in S, r \in R \Rightarrow a.r \in S$ and $r.a \in S$.

## SIMPLE RING :

A ring is said to be a simple ring if it has no non trivial proper ideal.

Example—

a) Every field is a simple ring.

b) In a ideal s of a ring R with contain a unit of R then S = R.

## PRINCIPAL IDEAL :

An ideal U of a ring R is said to be a principal ideal of R if U = <a> for some a in R.

Note—

Let R be a ring. The null ideal {0} is the smallest ideal of R containing the element 0. The null ideal {0} is a principal ideal of R.

## PRINCIPAL IDEAL RING :

A ring is said to be principal ideal ring if every ideal of the ring is a principal ideal.

Example—

a) The ring Z is a principal ideal ring.

b) The ring Zn is a principal ideal ring.

PRIME IDEAL :

In a ring R, an ideal $P \neq R$ is said to be a prime ideal if for a, b in R $ab \in P$ implies either $a \in P$ or $b \in P$.

Example—

a) The 2Z in the ring Z is a prime ideal.

MAXIMAL IDEAL :

An ideal M of a commutative ring R is called a maximal ideal of R, iff for any ideal U of R satisfying $M \subset U \subset R$.

Example—

a) In a ring Z the ideal 2Z is maximal ideal but 4Z is not maximal ideal in Z.

b) $R = c[0,1]$, $S = \{f \in R \,;\, f(1/2) = 0\}$ then S is a maximal ideal.

Note—

a) Every ideal of the ring Zn is a principle ideal.

b) A maximal ideal in a commutative ring without unity may not be a prime ideal.

PRINCIPAL IDEAL DOMAIN :

Principal ideal ring which is integral domain is called principal ideal domain.

i.e, Principal Ideal Ring + Integral Domain = Principal Ideal Domain.

Example—

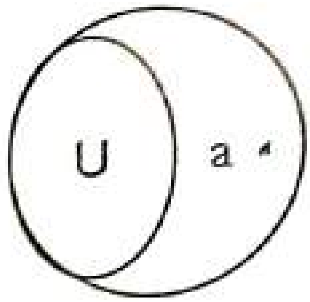
a) The ring Z is a Principal ideal domain.

b) Polynomial ring Z[x] is not a principal ideal domain.

## QUOTIENT RING :

Let R is a ring and U is an ideal R, since R is a ring with respect to '+' and ' . '

Let us consider a set R/U = {a + U} , a ∈ R




We define, addition and multiplication

$(a + U) + (b + U) = (a + b) + U$

$(a + U) * (b + U) = ab + U \quad \forall a, b \in R$

this ring R/U is called quotient ring.

Notes--

i. If U = < 0 > then R/U = R if U= R then R/U = < 0 >. Then zero element in the quotient ring R/U is U.

ii. If R be a commutative ring, then the quotient ring R/U is also commutative ring.

iii. If R be a ring with unity I and U be a proper ideal of R, then the quotient ring R/U is a ring with unity, I + U being the unity.

Results—

I. If a commutative ring R with unity, an ideal P is a prime ideal iff the quotient ring R/P is an integral domain.

II. If a commutative ring r with unity, an ideal M is a maximal ideal iff the quotient ring R/M is a field.

III. Every maximal ideal in a commutative ring with unity is a prime ideal. The converse may or may-not be hold.

# CONCLUSION

This project was a great way to help myself realize some
ings that I before. Like how much work I actually put into my
ssignments and how much I actually understand the work that
s put in front of me . Usually we don't look at the work after we
go home because, do it at home and you have friends around
you at college to help divide the work. This is how student's
minds work, and actually how I think about it sometimes too.
Sometimes I don't understand the importance of teachers
having us do these projects that see, to take a lifetime, but then
at the end of the day when it's all over I finally grasp the
concept and the idea of the whole thing and why they make us
do it in the first place. Because they want us to learn the
importance of what we do in class or what we have learned and
make sure we don't leave this college with a miss understanding.

# REFERENCE

I have done it by myself with the help of some books which are given in their reference


- HIGHER ALGEBRA ( S.K. Mapa )
- ABSTRACT ALGEBRA ( Charles Lanski )
- ADVANCE ALGEBRA ( Madhumangal pal )